LINEAR DYNAMIC SYSTEM IDENTIFICATION IN THE FREQUENCY DOMAIN USING FRACTIONAL DERIVATIVES

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Abstract

This paper presents a study of the Fourier transform method for parameter identification of a linear dynamic system in the frequency domain using fractional differential equations. Fundamental definitions of fractional differential equations are briefly outlined. The Fourier transform method of identification and their algorithms are generalized so that they include fractional derivatives and integrals.

Keywords: fractional differential equations, fractional differential systems, Fourier transform method, identification methods

1. Introduction

Nowadays, the rapid development of computer science, signal processing, material science, *etc*., enables us to use more precise and more sophisticated methods for data processing that were too complicated to be conducted in the past. One of them is the "new" mathematical theory of the fractional calculus, which can be applied in identification of parameters in the system.

The theorem of fractional calculus has been known since the 19th century. Until the 1970s and 1980s, fractional calculus was used exclusively in pure mathematics. Since that time, fractional calculus has been used in other applied technologies [1, 2].

In classical methods of identification, models based on elements of integer degree were used. This proves that there are systems and materials for which the traditional computational methods may not be adequate. In this case, methods of identification using extended fractional elements will be more precise and will reflect the real physical phenomena. A very good example of this is the piezoelectric PVDF [4] and modified bismuth oxides material [11, 12] which shows the importance of applying fractional elements in the process of identification. Electrical properties of these materials such as impedance (real and imaginary part vs. ω [7] including the influence of temperature) are difficult to describe by traditional mathematical equations. It is clear from the research of the authors of these materials that applying fractional elements to describe their electrical structure and fractional calculus to identify their parameters gives much simpler mathematical dependencies. They describe occurring physical phenomena more accurately than classical methods.

This is why the article below shows the generalized Fourier transform method, on the basis of which we carried out the process of parameter identification.

2. Background

Below, fundamental definitions of fractional calculus are shown: fractional integral and fractional derivative of Riemann-Liouville [8] will be used (in the next section) in connection with the problem of identification. All the problems presented in this paper concern the interval of integration $(0, t)$.

Definition 1 of fractional integral

Let Re $z > 0$ and let $f(t)$ for $t > 0$ be the piecewise continuous on $J' = (0, \infty)$ and integrable on any finite subinterval $J = [0, \infty)$, which can be defined as the fractional integral of f(t) of the order z:

$$D^{-z} f(t) = \frac{1}{\Gamma(z)} \int_0^t (t - \varepsilon)^{z-1} f(\varepsilon) d\varepsilon, \tag{2.1}$$

where $\Gamma(z)$ is the gamma function [8]. It is easy to see that for $z = n$ where n is the integer, we obtain an ordinary integral of $f(t)$.

Definition 2 of fractional derivative

Let $\mu > 0$ and m be the smallest integer that exceeds μ and $z = m - \mu > 0$. Then the fractional derivative of $f(t)$ of the order μ for $t > 0$ is defined as:

$$D^\mu f(t) = D^m [D^{-z} f(t)]. \tag{2.2}$$

The formula (2.2) results from m-times differentiation of the formula (2.1), so as to obtain μ fractional order of derivative. If $m = \mu$, then $z = 0$ and formula (2.2) becomes the simple identity. Instead, if μ is the positive integer which equals p, then the derivative $D^p f(t)$ can exist for $t > 0$, even if $f(t)$ is not of class of functions described in Definition 1 [8]. For example: let $f(t) = t^{-1}$. In case of f(t) having p continuous derivative on J, this function satisfies definition 1. We can see this clearly from the expression given below:

$$D^p f(t) = D^{p+1} \int_o^t f(\varepsilon) d\varepsilon = D^p f(t). \tag{2.3}$$

Formula (2.2) agrees with the classical definition of the ordinary derivative [8].

3. Fourier transform method for identification

In this paper an identification method of system parameters based on Fourier transform for any defined intervals and for all conditions in definitions 1 and 2 is considered. Let $u(t)$ and $y(t)$ be input and output signals of the system shown in Fig. 3.1.
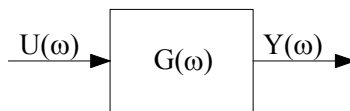

Fig. 3.1. Ideal system with a frequency input $U(\omega)$ and frequency output $Y(\omega)$.

Their Fourier transforms are further denoted by $U(\omega)$ and $Y(\omega)$, respectively. Then $G(\omega) = \dfrac{Y(\omega)}{U(\omega)}$ be the transfer function of this system and $\hat{G}(\omega)$ be the model to be identified.

Therefore when approximation for empirical results of input and output is used, the transfer function of the linear system $\hat{G}(\omega)$ with the known degrees of polynomials of the input and output functions can be obtained. In this way, the transfer function can contain an error. For a better presentation of the problem we disregard the noise in the system.

The transfer function of the dynamic linear system could be expressed as [3, 10]:

$$\hat{G}(\omega) = \frac{b_0 + b_1(j\omega) + \ldots + b_m(j\omega)^m}{a_0 + a_1(j\omega) + \ldots + a_n(j\omega)^n},$$ (3.1)

where $a_0 = 1$.

The transfer function described by (3.1) consists of the polynomials of the integer degrees. Based on the above information, we can determine the equation (3.1) coefficients using minimization of mean square error between the approximating transfer function and the true transfer function of the system.

4. Foundation of identification algorithm

In this section the generalized classical Fourier method for fractional elements is presented. It applies to the systems which meet the requirements of definition 1 and 2 of fractional integral and fractional derivative, as well as the conditions of Fourier transform.

Basic fractional systems, which can be easily applied to complicated systems, are under consideration. In order to analyze such systems, the fractional elements $(j\omega)^{\mu}$ were added into the transfer function (3.1) – in the numerator element $b_{\mu}(j\omega)^{\mu}$ and (or) in the denominator $a_{v}(j\omega)^{v}$, where μ and v are known fractional values. Then the extended form of the transfer function is as follows:

$$\hat{G}(\omega) = \frac{b_0 + b_{\mu}(j\omega)^{\mu} + b_1(j\omega) + \ldots + b_m(j\omega)^m}{a_0 + a_{v}(j\omega)^{v} + a_1(j\omega) + \ldots + a_n(j\omega)^n}.$$ (4.1)

where $a_0 = 1$ and fractional element is noted as:

$$(j\omega)^{\mu} = \omega^{\mu}\left(\cos\frac{\pi}{2}\mu + j\sin\frac{\pi}{2}\mu\right) = \omega^{\mu}\cos\frac{\pi}{2}\mu + j\omega^{\mu}\sin\frac{\pi}{2}\mu.$$ (4.2)

It must be noted that the fractional degree of order μ of element $(j\omega)^{\mu}$ can be larger than v in the assumed interval. However, by using several transformations which were shown in [9, 5], we can reduce the order of μ to the interval $[0,1]$. In the model describing the system it is possible to use more fractional elements, although we solve the problem in the same way as with one fractional element.

The expression (4.1) can otherwise be written as:

$$\hat{G}(\omega) = \frac{\alpha(\omega) + j\beta(\omega)}{p(\omega) + jq(\omega)} = \frac{P(\omega)}{Q(\omega)},$$ (4.3)

while the function $G(\omega)$ of the process is described as:

$$G(\omega) = R(\omega) + jI(\omega).$$ (4.4)

Comparing $G(\omega)$ and $\hat{G}(\omega)$ for values of frequency ω_i, the error in fitting $\varepsilon(\omega_i)$ is then:

$$\varepsilon(\omega_i) = G(\omega_i) - \hat{G}(\omega_i). \tag{4.5}$$

Substituting expression (4.3) for $\hat{G}(\omega)$ in the above equation, the error $\varepsilon(\omega_i)$ can be expressed as:

$$\varepsilon(\omega_i) = G(\omega_i) - \frac{P(\omega_i)}{Q(\omega_i)}. \tag{4.6}$$

The next step is the minimization of the error which results from the difference between the real transfer function and the transfer function based on approximation (interpolation) in measuring the results. We could use the least square fit to obtain unknown coefficients of transfer function $\hat{G}(\omega)$, but the problem is difficult when solved in this manner [6, 10]. However it is also possible to use the least square method modified by Levy [6].

Therefore, multiplying (4.6) by a polynomial $Q(\omega)$ we obtain:

$$Q(\omega_i)\varepsilon(\omega_i) = Q(\omega_i)G(\omega_i) - P(\omega_i). \tag{4.7}$$

For simplicity, the left side of (4.7) could be written as the sum of elements A and B:

$$Q(\omega_i)\varepsilon(\omega_i) = A(\omega_i) + jB(\omega_i). \tag{4.8}$$

The magnitude of this equation is:

$$\left|Q(\omega_i)\varepsilon(\omega_i)\right|^2 = A^2(\omega_i) + B^2(\omega_i). \tag{4.9}$$

The left side of (4.9) could be written as a mean square error [6]. Thus the error function E may be defined as the sum of the two elements A and B:

$$E = \sum_{i=1}^{k} \left[A^2(\omega_i) + B^2(\omega_i) \right]. \tag{4.10}$$

To simplify notations we skip ω_i like this $\hat{G}_i = \hat{G}(\omega_i)$, *etc.*

Comparing the two equations (4.7) and (4.8) with regard to their real and imaginary parts, element A could be written as a real part, but element B as an imaginary part of the right-hand side of equation (4.7).

For element A:

$$A_i = \mathrm{Re}\left[Q_i G_i - P_i \right]. \tag{4.11}$$

Replacing accordingly Q, G, P in (4.11), yields:

$$A_i = \mathrm{Re}\left[\left(p_i + jq_i \right)\left(R_i + jI_i \right) - \left(\alpha_i + j\beta_i \right) \right]. \tag{4.12}$$

Thus:

$$A_i = p_i R_i - q_i I_i - \alpha_i. \tag{4.13}$$

Realizing the same for element B_i, we obtain:

$$B_i = q_i R_i + p_i I_i - \beta_i. \tag{4.14}$$

Below, (4.13) and (4.14) are applied to (4.10) to yield:

$$E = \sum_{i=1}^{k}\left[\left(p_i R_i - q_i I_i - \alpha_i\right)^2 + \left(q_i R_i + p_i I_i - \beta_i\right)^2\right], \tag{4.15}$$

where the coefficients in (4.15) are:

$$
\begin{cases}
\alpha_i = b_0 + b_\mu \omega_i^\mu \cos\dfrac{\pi}{2}\mu - b_2 \omega_i^2 + b_4 \omega_i^4 - \ldots \\[2mm]
\beta_i = b_\mu \omega_i^\mu \sin\dfrac{\pi}{2}\mu + b_1 \omega - b_3 \omega_i^3 + b_5 \omega_i^5 - \ldots \\[2mm]
p_i = a_0 + a_\nu \omega_i^\nu \cos\dfrac{\pi}{2}\nu - a_2 \omega_i^2 + a_4 \omega_i^4 - \ldots \\[2mm]
q_i = a_\nu \omega_i^\nu \sin\dfrac{\pi}{2}\nu + a_1 \omega - a_3 \omega_i^3 + a_5 \omega_i^5 - \ldots
\end{cases}
\tag{4.16}
$$

To find the coefficients of the transfer function, we differentiate the error function in dependence on coefficients a_i and b_i and next the results are equated to zero. Then we obtain the system of equations (4.17):

$$
\begin{cases}
\dfrac{\partial E}{\partial b_0} = -2\sum_{i=1}^{k}\left(p_i R_i - q_i I_i - \alpha_i\right) = 0 \\[3mm]
\dfrac{\partial E}{\partial b_\mu} = -2\sum_{i=1}^{k}\omega_i^\mu\left[\left(p_i R_i - q_i I_i - \alpha_i\right)\left(\cos\dfrac{\pi}{2}\mu\right) + \left(q_i R_i + p_i I_i - \beta_i\right)\left(\sin\dfrac{\pi}{2}\mu\right)\right] = 0 \\[3mm]
\dfrac{\partial E}{\partial b_1} = -2\sum_{i=1}^{k}\omega_i\left(q_i R_i + p_i I_i - \beta_i\right) = 0 \\[2mm]
\vdots \\[2mm]
\dfrac{\partial E}{\partial a_\nu} = 2\sum_{i=1}^{k}\omega_i^\nu\left[\begin{array}{l}\left(p_i R_i - q_i I_i - \alpha_i\right)\left(R_i \cos\dfrac{\pi}{2}\nu - I_i \sin\dfrac{\pi}{2}\nu\right) + \\[2mm] \left(q_i R_i + p_i I_i - \beta_i\right)\left(R_i \sin\dfrac{\pi}{2}\nu + I_i \cos\dfrac{\pi}{2}\nu\right)\end{array}\right] = 0 \\[5mm]
\dfrac{\partial E}{\partial a_1} = 2\sum_{i=1}^{k}\omega_i\left[\left(p_i R_i - q_i I_i - \alpha_i\right)\left(-I_i\right) + \left(q_i R_i + p_i I_i - \beta_i\right)\left(R_i\right)\right] = 0
\end{cases}
\tag{4.17}
$$

Substituting (4.16) into (4.17) we can evaluate the coefficients of the transfer function. When Laplace transformation is used in identification, the same procedure can be applied as in the Fourier transform method, taking into account the conditions of definitions 1 and 2 regarding existing fractional integrals and derivatives and also the conditions for Laplace transformation.

5. The example

Below, an example presents the identification of parameters with the Fourier transform method with error minimization. We use the modified least square method for the system including fractional elements $(j\omega)^\mu$.

In this example it is assumed that the stimulated input signal and output signals of the system are known.

To simplify the transformations and calculations presented in this example, transfer function $\hat{G}$ is known (without known coefficients) and is close to the real transfer function G.

In this example we show a transfer function where fractional elements are presented in the numerator and denominator.

The example is chosen to be as simple as possible while retaining just sufficient complexity to illustrate the identification of parameters of the linear dynamic system transfer function containing fractional elements.

To demonstrate the identification procedure, Figs (5.1–2) are examined for frequencies in the following range:

$$\omega_i \in \left[10^{-2}, 10\right], \tag{5.1}$$
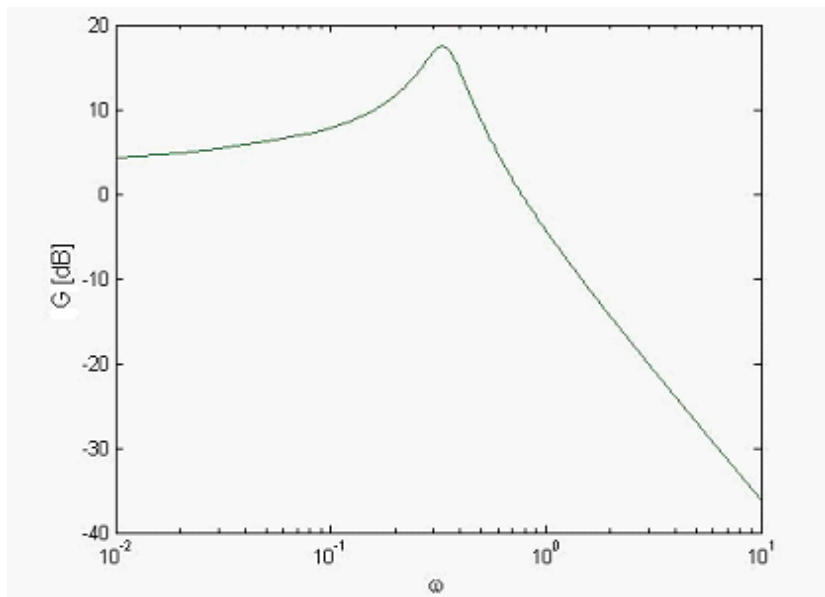
where: $i = 1, \ldots, k = 1000$

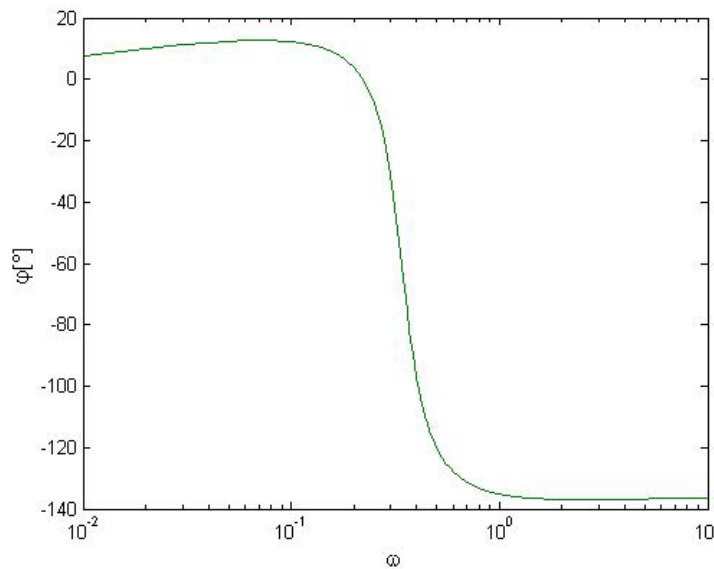Fig. 5.1. Dependence of magnitude $G(\omega)$.

Fig. 5.2. Dependence of phase $\phi(\omega)$.

The true transfer function of the system $G(\omega)$ is the sum of obtained measurement results of the imaginary $I(\omega)$ and real $R(\omega)$ forms.

Thus:

$$G(\omega) = R(\omega) + jI(\omega). \tag{5.2}$$

The approximated transfer function form $\hat{G}(\omega)$ is assumed to be as follows:

$$\hat{G}(\omega) = \frac{b_\mu (j\omega)^\mu + b_0}{a_2 (j\omega)^2 + a_v (j\omega)^v + 1}. \tag{5.3}$$

The values of fractional degree are assumed to be: $v = \frac{1}{3}$, $\mu = \frac{1}{2}$. To simplify the notation we used substitutions as below:

$$\begin{cases} C_\vartheta = \cos\left(\vartheta \frac{\pi}{2}\right) \\ S_\vartheta = \sin\left(\vartheta \frac{\pi}{2}\right) \end{cases}, \tag{5.4}$$

where: $\vartheta = v$ or $\vartheta = \mu$.

The mean square error between the approximated transfer function $\hat{G}(\omega)$ and the true transfer function $G(\omega)$, is expressed on the basis of equation (4.15), which is presented as (4.10). Using transformation of equations (4.16), the coefficients α_i, β_i, p_i, q_i, have the following expansions:

$$\begin{cases} \alpha_i = b_0 + C_\mu \omega_i^\mu b_v \\ \beta_i = S_\mu \omega_i^\mu b_\mu \\ p_i = -\omega_i^2 a_2 + C_v \omega_i^v a_v + 1 \\ q_i = S_v \omega_i^v a_v \end{cases}. \tag{5.5}$$

Therefore, the partial derivative of error E with respect to each of the coefficients of the transfer function could be written according to a system of equations (4.17) as:

$$\begin{cases} \dfrac{\partial E}{\partial b_0} = -2\sum_{i=1}^{k} A_i = 0 \\[2mm] \dfrac{\partial E}{\partial b_\mu} = -2\sum_{i=1}^{k} \omega_i^\mu \left[A_i C_\mu + B_i S_\mu \right] = 0 \\[2mm] \dfrac{\partial E}{\partial a_v} = 2\sum_{i=1}^{k} \omega_i^v \left[A_i \left(R_i C_v - I_i S_v \right) + B_i \left(R_i S_v + I_i C_v \right) \right] = 0 \\[2mm] \dfrac{\partial E}{\partial a_2} = -2\sum_{i=1}^{k} \omega_i^2 \left[A_i R_i + B_i I_i \right] = 0 \end{cases}. \tag{5.6}$$

Introducing the notation as below:

$$F_v = \left(R_i C_v - I_i S_v \right) \omega_i^v$$

$$F_\mu = \left(R_i C_\mu - I_i S_\mu \right) \omega_i^\mu$$

$$L_\mu = \left(R_i C_\mu + I_i S_\mu \right) \omega_i^\mu .$$

$$L_v = \left(R_i S_v + I_i C_v \right) \omega_i^v$$ (5.7)

$$K = \left(R_i^2 + I_i^2 \right) \omega_i^2$$

and substituting expressions (5.5) for α_i, β_i, p_i, q_i in the system of equations (5.6), a new system of equations is obtained:

$$\sum_{i=1}^{k} \begin{cases} -R_i \omega_i^2 a_2 + F_v a_v - C_\mu \omega_i^\mu b_\mu - b_0 = -R_i \\ -L_\mu \omega^2 a_2 + \left(F_v C_\mu + L_v S_\mu \right) \omega_i^\mu a_v - \omega_i^{2\mu} b_\mu - C_\mu \omega_i^\mu b_0 = -L_\mu \\ -C_v \omega_i^v K a_2 + K \omega_i^{2v-2} a_v - \left(C_\mu F_v + S_\mu L_v \right) \omega_i^\mu b_\mu - F_v b_0 = -K C_v \omega^{v-2} \\ -K \omega_i^2 a_2 + K C_v \omega_i^v a_v - L_\mu \omega^2 b_\mu - R_i \omega_i^2 b_0 = -K \end{cases} .$$ (5.8)

In this way a system of four equations with four unknown coefficients is obtained. Substituting the values for the above system of equations and calculating the sums of the k elements, we are given the matrix which can be written as:

$$K \cdot M = N,$$ (5.9)

where:

$$K = \begin{bmatrix} 0.7684 & -0.0526 & -1.4918 & -1 \\ 2.5281 & -0.1800 & -5.0050 & -1.4918 \\ -0.1755 & 0.5514 & 0.1800 & 0.0526 \\ -1.2923 & 0.1755 & 2.5281 & 0.7684 \end{bmatrix} .$$ (5.10)

$$M^T = \begin{bmatrix} a_2 & a_v & b_\mu & b_0 \end{bmatrix},$$ (5.11)

$$N^T = \begin{bmatrix} -0.0214 & 0.1806 & -0.6759 & -0.2086 \end{bmatrix}.$$ (5.12)

Since the system of equations has only one solution, we had to assume a value of the coefficients which we were looking for in order to carry out the correct computation. The computed results were compared to the actual results of experiments $I(\omega)$ and $R(\omega)$, correcting for the assumed value of the parameters. It was verified by applying the above procedure to other parameters. In this manner we computed our coefficients:

$$\begin{cases} a_2 = 29.40 \\ a_\mu = 3.65 \\ b_v = 14.20 \\ b_0 = 1.62 \end{cases} .$$ (5.13)

Calculating the maximal difference between magnitudes $dif|G|$ or phase $dif|\varphi|$ of the true transfer function G and the approximating transfer function $\hat{G}$, we obtain:

$$dif |G| = \max \left| G_i - \hat{G}_i \right| = 4.6 \cdot 10^{-11}$$ (5.14)

and

$$dif\,\phi = \max\left|\phi_i - \hat{\phi}_i\right| = 2.3 \cdot 10^{-10}. \tag{5.15}$$

The obtained values of $dif\,|G|$ and $dif\,\phi$ are very small and they show that the assumed model is close to the examined system.

6. Conclusion

A study of the Fourier transform method for parameter identification of a linear dynamic system in the frequency domain using fractional differential equations has been presented in this article. It must be pointed out that fractional elements introduce complex interdependencies in solving system equations. It has been shown how to calculate the coefficients of the approximating function of finite order using as a criterion the minimum of the mean square error between the true and approximating frequency responses.

Considering more complicated models with fractional elements, numerical problems can be obtained, resulting from the transformation of matrixes including forms of fractional elements. Therefore, it is advisable to assume one or more desired (computed) parameters.

It is of main interest to propose the fractional system for modelling where conventional integration/ derivation is replaced by a fractional one due to precise reflection of the real systems what was presented in practice in [5, 11, 12].

References

[1] M. Axtell, E.M. Bise: "Fractional calculus applications in control systems". *Proc. IEEE Nat. Aerospace and Electronics Conf.*, 1990, pp. 563–566.

[2] R.L. Bagley, P.J. Torvik: "On the appearance of the fractional derivative in the behavior of real materials". *J. Appl. Mech.*, no. 51,1984, pp. 294–298.

[3] P. Eykhoff: *System identification. Parameter and Sate Estimation*. John Wiley and Sons Ltd., London, 1974.

[4] T. Janiczek: "Analysis of PVDF transducer signals stimulated by mechanical tension". *Journal of Electrostatics*, vol. 51–52, 2001, pp. 167–172.

[5] T. Janiczek: *Models of systems described by fractional differential equations and basic algorithms of their identification*. Preprint of Ph.D., Wroclaw University of Technology, 2003.

[6] E.C. Levy: "Complex curve fitting". *IRE Trans. Aut. Contr.*, no. 4, 1959, pp. 37–43.

[7] J.R. MacDonald, W.R. Kenan: *Impedance Spectroscopy: Emphasis Solid Materials and Systems*. Wiley-Interscience, 1987.

[8] K.S. Miller, B. Ross: *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley&Sons Inc. 1993.

[9] I. Podlubny: *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[10] Y. Sawaragi, T. Soeda, T. Nakamizo: *Classical Methods and time series estimation. Trends And Progress In System Identification*. Ed. by Pieter Eykhoff, Pergamon Press, Oxford, 1981.

[11] T. Janiczek, D. Nowak-Woźny, W. Mielcarek, K. Prociow. "Equivalent model of modified bismuth oxides described by fractional derivatives". *The Fourth China International Conference on High-Performance Ceramics*, Oct., 2005.

[12] D. Nowak-Woźny, T. Janiczek, W. Mielcarek, J.B. Gajewski: Fractional electrical model for modified bismuth oxide. *Journal of Electrostatics*, no. 67, 2009, pp. 18–21.